Digital Fourier transforms

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Version 1.0

In the two previous lectures, we have seen how to transform an analog signal into a digital one with sampling and how to transform a digital signal into an analog one with interpolation. We use these transforms to introduce the digital Fourier transforms: the Discrete-Time Fourier Transform (DTFT) corresponding to the Fourier transform, and the Discrete Fourier Transform (DFT) corresponding to the Fourier series.

1 Discrete-time Fourier transform

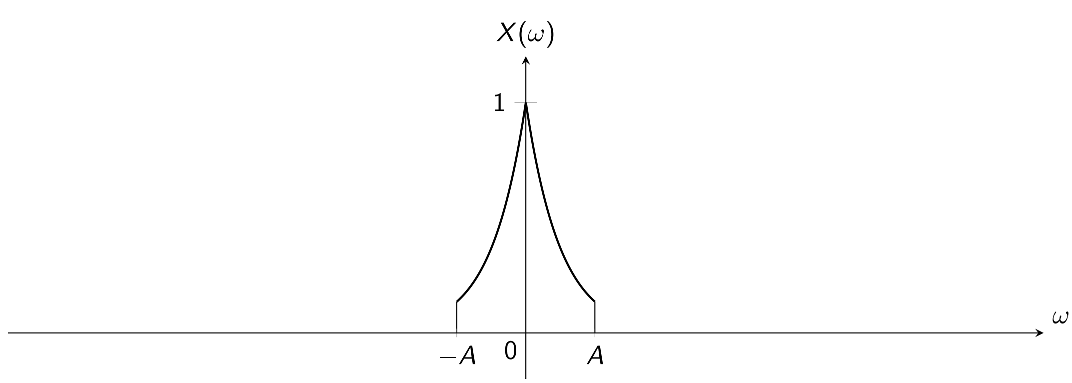
1.1 Definition

Let x be a digital signal and $x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\delta(t - nT_s)$ the corresponding sampled signal. By linearity of the Fourier transform and by definition of the Fourier transform of a shifted Dirac delta function, the spectrum X_s of signal x_s is written:

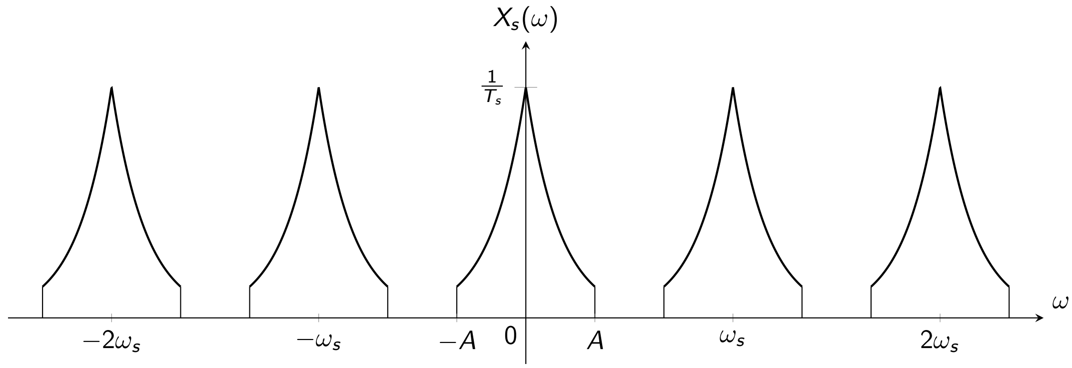
$$\forall \omega \in \mathbb{R} \qquad X_s(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-inT_s\omega}$$

This spectrum depends on the sampling period T_s which does not appears in the equations with digital signals. Introducing the **normalized frequency** $\nu = \omega T_s = \frac{\omega}{f_s}$, we can define the discrete-time Fourier transform.

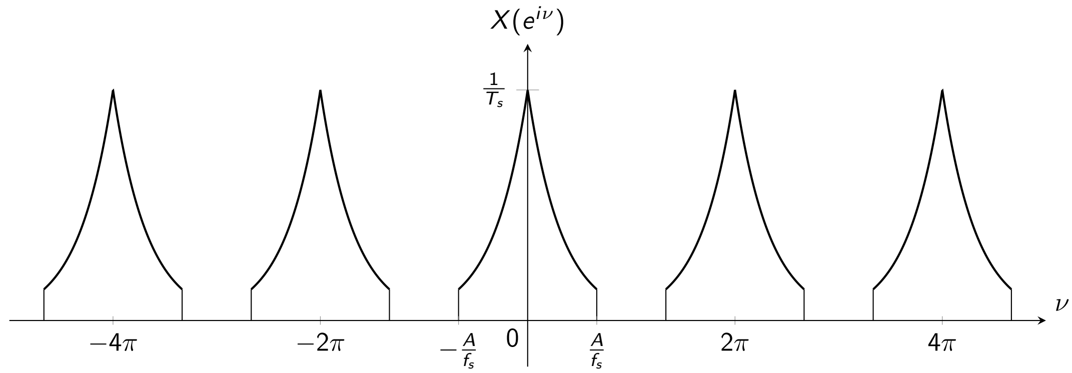
> **Definition 1.1 (Discrete-time Fourier transform)**
> The **discrete-time Fourier transform** (DTFT) is an application from $\mathcal{F}(\mathbb{Z}, \mathbb{C})$ to $\mathcal{F}(\mathbb{R}, \mathbb{C})$ which maps a digital signal x to the analog function $X = \mathcal{F}(x)$ defined by:
>
> $$\forall \nu \in \mathbb{R} \qquad X(e^{i\nu}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-in\nu} = \sum_{n=-\infty}^{+\infty} x[n]\left(e^{i\nu}\right)^{-n}$$



Spectrum of the analog signal



Spectrum of the sampled signal



Spectrum of the digital signal

We notice on this figure and we can prove from the definition that the DTFT of a digital signal is periodic with period 2π. In addition, its modulus is obtained by dividing the modulus of the spectrum of the original analog signal by T_s.

1.2 Inverse DTFT

Now we construct the inverse discrete-time Fourier transform (IDTFT), i.e. we express the samples of the digital signal x from its DTFT $X(e^{i\nu})$. Consider a digital signal x obtained by sampling an analog signal also denoted x with spectrum X, associated with the sampled signal x_s with spectrum X_s. Using the definition of the analog inverse Fourier transform, we have:

$$\forall t \in \mathbb{R} \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{i\omega t} d\omega \qquad \text{that implies} \qquad \forall n \in \mathbb{Z} \quad x[n] = x(nT_s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{i\omega n T_s} d\omega$$

As shown on the previous figures, if we want to recover $X(\omega)$ from DTFT $X(e^{i\nu})$, we have to apply an ideal lowpass filter with cutoff frequency $\omega_{co} = \frac{\omega_s}{2}$ followed by an amplifier of factor T_s, i.e. the system with frequency response $H(\omega) = T_s R_{[-\frac{\omega_s}{2}, \frac{\omega_s}{2}]}(\omega)$. Thus we have for any $\omega \in \mathbb{R}$,

$$X(\omega) = X_s(\omega) H(\omega) = T_s X_s(\omega) R_{[-\frac{\omega_s}{2}, \frac{\omega_s}{2}]}(\omega)$$

Applying the change of variable $\nu = \omega T_s$, we get:

$$x[n] = \frac{T_s}{2\pi} \int_{-\frac{\omega_s}{2}}^{\frac{\omega_e}{2}} X_s(\omega) e^{i\omega n T_s} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{i\nu}) e^{i\nu n} d\nu$$

Definition 1.2 (Inverse discrete-time Fourier transform)
Consider a digital signal x whose DTFT is $X(e^{i\nu})$. Then

$$\forall n \in \mathbb{Z} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{i\nu})e^{i\nu n}d\nu$$

1.3 Properties

Proposition 1.1
Discrete-time Fourier transform satisfies the following properties:

(i) linearity: for any two signals x and y and two scalars α and β, $\mathcal{F}(\alpha x + \beta y) = \alpha\mathcal{F}(x) + \beta\mathcal{F}(y)$;

(ii) symmetry: for any signal x, if we denote $\tilde{x} : n \mapsto x[-n]$, then $\mathcal{F}(\tilde{x}) = \widetilde{\mathcal{F}(x)} = \mathcal{F}(x^*)^*$;

(iii) decimation: for any $K \in \mathbb{N}^*$ and any digital signal x, if we define the decimated signal $x_K : n \mapsto x[Kn]$ and X_K its DTFT, then

$$\forall \nu \in \mathbb{R} \qquad X_K(e^{i\nu}) = \frac{1}{K}\sum_{k=0}^{K-1} X\left(\exp\left(i\frac{\nu - k2\pi}{K}\right)\right)$$

(iv) convolution: for any two signals x and y, $\mathcal{F}(x * y) = \mathcal{F}(x)\mathcal{F}(y)$;

(v) time-shift: for any $k \in \mathbb{Z}$ and any digital signal x, $\mathcal{F}(\tau_k(x))\left(e^{i\nu}\right) = e^{-ik\nu}\mathcal{F}(x)\left(e^{i\nu}\right)$;

(vi) time differentiation: for any digital signal x, the DTFT of the digital derivative $x' : n \mapsto x[n] - x[n-1]$ is $\mathcal{F}(x')\left(e^{i\nu}\right) = (1 - e^{-i\nu})\mathcal{F}(x)\left(e^{i\nu}\right)$;

(vii) frequency differentiation: for any signal x, if we set $y : n \mapsto -inx[n]$, then $\mathcal{F}(y) = (\mathcal{F}(x))'$;

(viii) multiplication: for any two digital signals x and y, $\mathcal{F}(xy) = \mathcal{F}(x) \otimes \mathcal{F}(y)$;

(ix) multiplication by a complex exponential: for any $a \in \mathbb{R}$, $\mathcal{F}(e^{ina}x) = \tau_a(\mathcal{F}(x))$.

PROOF : (i) The linearity of the TDFT is deduced from the linearity of the sum.
(ii) By the change of variable $n \mapsto -n$, we obtain, for any $\nu \in \mathbb{R}$,

$$\mathcal{F}(\tilde{x})\left(e^{i\nu}\right) = \sum_{n=-\infty}^{+\infty} x[-n]e^{-in\nu} = \sum_{n=-\infty}^{+\infty} x[n]e^{-in(-\nu)} = \mathcal{F}(x)\left(e^{i(-\nu)}\right) = \left(\sum_{n=-\infty}^{+\infty} x[n]^* e^{-in\nu}\right)^* = \mathcal{F}(x^*)^*\left(e^{i\nu}\right)$$

(iii) To prove this property, we need to go back to sampling. Let an analog signal x and a sampling period T_s producing a digital signal also denoted x, corresponding to the sampled signal x_s. Signal x_K results from the sampling of analog signal x with the sampling period $T_s' = KT_s$, i.e. the frequency $\omega_s' = \frac{\omega_s}{K}$. We associate x_K with the sampled signal $x_{s,K}$. Using Poisson summation formula, we can express the spectra of the sampled signals:

$$X_s(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} X(\omega - n\omega_s) \qquad \text{and} \qquad X_{s,K}(\omega) = \frac{1}{KT_s}\sum_{n=-\infty}^{+\infty} X\left(\omega - \frac{n\omega_s}{K}\right)$$

Using bijection $\mathbb{Z} \times [\![0, K-1]\!] \to \mathbb{Z} \quad (n, k) \mapsto nK + k$, we can write

$$X_{s,K}(\omega) = \frac{1}{KT_s} \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{K-1} X\left(\omega - \frac{k\omega_s}{K} - n\omega_s\right) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} X\left(\left(\omega - \frac{k\omega_s}{K}\right) - n\omega_s\right) = \frac{1}{K} \sum_{k=0}^{K-1} X_s\left(\omega - \frac{k\omega_s}{K}\right)$$

For the DTFT of x_K, the normalized frequency is $\nu = \omega K T_s$, so that

$$X_K(e^{i\nu}) = X_{s,K}\left(\frac{\nu}{KT_s}\right) = \frac{1}{K} \sum_{k=0}^{K-1} X_s\left(\frac{(\nu/T_s) - k\omega_s}{K}\right) = \frac{1}{K} \sum_{k=0}^{K-1} X\left(\exp\left(i\frac{\nu - k2\pi}{K}\right)\right)$$

(iv) For any $\nu \in \mathbb{R}$,

$$\begin{aligned}\mathcal{F}(x * y)\left(e^{i\nu}\right) &= \sum_{n=-\infty}^{+\infty} (x * y)[n] e^{-in\nu} = \sum_{n=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k] e^{-ik\nu} y[n-k] e^{-i(n-k)\nu} \\ &= \left(\sum_{n=-\infty}^{+\infty} x[n] e^{-in\nu}\right)\left(\sum_{m=-\infty}^{+\infty} y[m] e^{-im\nu}\right) = \mathcal{F}(x)\left(e^{i\nu}\right)\mathcal{F}(y)\left(e^{i\nu}\right)\end{aligned}$$

(v) For any $\nu \in \mathbb{R}$ and any $k \in \mathbb{Z}$,

$$\mathcal{F}(\tau_k(x))\left(e^{i\nu}\right) = \sum_{n=-\infty}^{+\infty} x[n-k] e^{-in\nu} = e^{-ik\nu} \sum_{n=-\infty}^{+\infty} x[n] e^{-in\nu} = e^{-ik\nu} \mathcal{F}(x)\left(e^{i\nu}\right)$$

(vi) The digital derivative can be written $x' = x - \tau_1(x)$. Using properties (i) and (v), we find, for any $\nu \in \mathbb{R}$,

$$\mathcal{F}(x')\left(e^{i\nu}\right) = \mathcal{F}(x)\left(e^{i\nu}\right) - \mathcal{F}(\tau_1(x))\left(e^{i\nu}\right) = (1 - e^{-i\nu})\mathcal{F}(x)\left(e^{i\nu}\right)$$

(vii) For any $\nu \in \mathbb{R}$,

$$\mathcal{F}(y)\left(e^{i\nu}\right) = \sum_{n=-\infty}^{+\infty} y[n] e^{-in\nu} = \left(\sum_{n=-\infty}^{+\infty} x[n] e^{-in\nu}\right)' = \sum_{n=-\infty}^{+\infty} (-in) x[n] e^{-in\nu}$$

thus by identification, for any $n \in \mathbb{N}$, $y[n] = -inx[n]$.
(viii) Let two digital signals x and y with respective DTFTs X and Y. Let $Z = X \otimes Y$ and z the corresponding digital signal. We are using the circular convolution here because DTFTs X and Y are both periodic with period 2π. Then for any $n \in \mathbb{Z}$,

$$z[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (X \otimes Y)(e^{i\nu}) e^{i\nu n} d\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{iu}) Y(e^{i(\nu-u)}) du\right) e^{i\nu n} d\nu$$

By the change of variable $(u, \nu) \mapsto (u, \nu - u)$ and by Fubini's theorem:

$$\begin{aligned}z[n] &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(e^{iu}) e^{iun} Y(e^{i(\nu-u)}) e^{i(\nu-u)n} du d\nu = \frac{1}{4\pi^2} \left(\int_{-\pi}^{\pi} X(e^{i\nu}) e^{i\nu n} d\nu\right)\left(\int_{-\pi}^{\pi} Y(e^{i\nu}) e^{i\nu n} d\nu\right) \\ &= x[n] y[n]\end{aligned}$$

Finally, $\mathcal{F}(xy) = \mathcal{F}(z) = \mathcal{F}(x) \otimes \mathcal{F}(y)$.
(ix) Let $a \in \mathbb{R}$. Then, for any $\nu \in \mathbb{R}$,

$$\mathcal{F}(e^{ina} x)\left(e^{i\nu}\right) = \sum_{n=-\infty}^{+\infty} e^{ina} x[n] e^{-in\nu} = \sum_{n=-\infty}^{+\infty} x[n] e^{-in(\nu-a)} = \mathcal{F}(x)\left(e^{-i(\nu-a)}\right) \qquad \blacksquare$$

Theorem 1.2 (Plancherel's identity)
Let a digital square summable signal x and its DTFT $X = \mathcal{F}(x)$. Then

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{i\nu})|^2 d\nu$$

Proof : We recognize on the left side the energy of signal x:

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = E(x) = \gamma_x[0] = (x * \tilde{x})[0]$$

with $\tilde{x} : n \mapsto x^*[-n]$. We set $y = (x * \tilde{x})$ and $Y = \mathcal{F}(y)$ its DTFT, so that

$$\forall n \in \mathbb{Z} \qquad y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{i\nu}) e^{i\nu n} d\nu \qquad \text{and} \qquad E(x) = y[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{i\nu}) d\nu$$

Applying property (ii) of Proposition 1.1, we can write $\mathcal{F}(\tilde{x}) = \mathcal{F}(x)^*$. Thus we have

$$Y(e^{i\nu}) = \mathcal{F}(x * \tilde{x})(e^{i\nu}) = \mathcal{F}(x)(e^{i\nu})\mathcal{F}(\tilde{x})(e^{i\nu}) = X(e^{i\nu})X^*(e^{i\nu}) = |X(e^{i\nu})|^2$$

which yields

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = y[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{i\nu}) d\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{i\nu})|^2 d\nu \qquad \blacksquare$$

2 Discrete Fourier transform

Now we introduce the Fourier transform for digital periodic signals, i.e. the digital version of Fourier series. First, we define this new transform based on the Fourier transform of the corresponding sampled signal, then we exhibit an orthonormal basis for digital periodic signals confirming the first approach.

2.1 First approach - definition

Let a digital periodic signal $x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ with period N. Let $x_s(t) = \sum_{n=-\infty}^{+\infty} x[n]\delta(t - nT_s)$ the sampled signal associated with x. From the periodicity of x, this signal x_s can be written:

$$x_s(t) = \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{+\infty} x[k + \ell N]\delta\left(t - (k + \ell N)T_s\right) = \sum_{k=0}^{N-1} x[k] \sum_{\ell=-\infty}^{+\infty} \delta\left(t - \ell N T_s - kT_s\right) = \sum_{n=0}^{N-1} x[n]\tau_{nT_s}(p_{NT_s})(t)$$

Using the linearity of the analog Fourier transform and Poisson summation formula, we can write the spectrum x_s for any $\omega \in \mathbb{R}$:

$$\begin{aligned} X_s(\omega) &= \sum_{n=0}^{N-1} x[n]\mathcal{F}\left(\tau_{nT_s}(p_{NT_s})\right)(\omega) = \sum_{n=0}^{N-1} x[n]e^{-i\omega nT_s}\mathcal{F}\left(p_{NT_s}\right)(\omega) \\ &= \frac{\omega_s}{N}\sum_{n=0}^{N-1} x[n]e^{-i\omega nT_s}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\frac{\omega_s}{N}\right) = \frac{\omega_s}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\frac{\omega_s}{N}\right)\sum_{n=0}^{N-1} x[n]e^{-i\omega nT_s} \\ &= \frac{\omega_s}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\frac{\omega_s}{N}\right)\sum_{n=0}^{N-1} x[n]e^{-ik\frac{\omega_s}{N}nT_s} = \frac{\omega_s}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\frac{\omega_s}{N}\right)\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn} \\ &= \frac{\omega_s}{N}\sum_{k=-\infty}^{+\infty} X[k]\delta\left(\omega - k\frac{\omega_s}{N}\right) \end{aligned}$$

with $X[k] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}$.

Remarks:

- We deduce from this computation that the Fourier transform of the sampled signal, thus the DTFT of the corresponding signal, is a sum of Dirac delta functions. This is consistent with the fact that the Fourier transform of an analog periodic signal is a sum of Dirac delta functions.
- We discarded the factor $\frac{\omega_s}{N}$ in the definition of $X[k]$. We give the reason for this removal with the second approach.
- Since x is periodic with period N, we only keep samples $x[0], \dots, x[N-1]$ in the expression of $X[k]$.

Definition 2.1 (Discrete Fourier Transform)
The **Discrete Fourier Transform** (DFT) is an application from $\mathcal{F}(\mathbb{Z}/N\mathbb{Z}, \mathbb{C})$ to $\mathcal{F}(\mathbb{Z}, \mathbb{C})$ which maps a time-limited digital signal x of length N to the digital signal X defined by:

$$\forall k \in \mathbb{Z} \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}$$

Proposition 2.1
The DFT X of a digital periodic signal with period N is also a digital periodic signal with period N: for any $k \in \mathbb{Z}$, $X[k+N] = X[k]$.

Proof : For any $k \in \mathbb{Z}$,

$$X[k+N] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}e^{-i\frac{2\pi}{N}Nn} = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}\left(e^{-i2\pi}\right)^n = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn} = X[k]$$ ∎

2.2 Second approach - inverse DFT

Now we discuss our second approach of the DFT by introducing an orthonormal basis for digital periodic signals.

Proposition 2.2

For any $j \in [\![0, N-1]\!]$, we define the following digital periodic signal $e_j \in \mathcal{F}_N(\mathbb{Z}, \mathbb{R})$:

$$\forall n \in \mathbb{Z} \qquad e_j[n] = \begin{cases} 1 & \text{if } n = j \mod N \\ 0 & \text{otherwise} \end{cases}$$

The set $(e_j)_{j \in [\![0,N-1]\!]}$ is a basis of vector space $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ whose dimension is then N.

PROOF : It is clear from the definition of signals e_j that any signal $x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ can be written $x = \sum_{j=1}^{N-1} x[j] e_j$, so that $(e_j)_{j \in [\![0,N-1]\!]}$ is a generating set of $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$. Let $(\lambda_0, \dots, \lambda_{N-1}) \in \mathbb{C}^N$ such that $\lambda_0 e_0 + \dots \lambda_{N-1} e_{N-1} = 0$, i.e. for any $n \in \mathbb{Z}$, $\lambda_0 e_0[n] + \dots \lambda_{N-1} e_{N-1}[n] = 0$. Taking $n = j$ for all $j \in [\![0, N-1]\!]$, we get $\lambda_j = 0$, thus $\lambda_0 = \dots = \lambda_{N-1} = 0$. Thereby, $(e_j)_{j \in [\![0,N-1]\!]}$ is linearly independent thus it is a basis of $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$. ∎

Proposition 2.3 (Inverse Discrete Fourier Transform)

The set of exponentials $\left(e^{i\frac{2\pi}{N}kn}\right)_{k \in [\![0,N-1]\!]}$ is an orthonormal basis of $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$, so that the digital signal $x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ with DFT X can be written:

$$\forall n \in \mathbb{Z} \qquad x[n] = \sum_{k=0}^{N-1} \langle x, e^{i\frac{2\pi}{N}kn} \rangle_N e^{i\frac{2\pi}{N}kn} = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi}{N}kn}$$

PROOF : For any $k \in [\![0, N-1]\!]$,

$$\|e^{i\frac{2\pi}{N}kn}\|_N^2 = \langle e^{i\frac{2\pi}{N}kn}, e^{i\frac{2\pi}{N}kn} \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}kn} \overline{e^{i\frac{2\pi}{N}kn}} = \frac{1}{N} \sum_{n=0}^{N-1} 1 = \frac{N}{N} = 1$$

For any $(k, \ell) \in [\![0, N-1]\!]^2$ such that $k \neq \ell$,

$$\langle e^{i\frac{2\pi}{N}kn}, e^{i\frac{2\pi}{N}\ell n} \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}kn} \overline{e^{i\frac{2\pi}{N}\ell n}} = \frac{1}{N} \sum_{n=0}^{N-1} e^{i\frac{2\pi}{N}(k-\ell)n} = \frac{1}{N} \frac{1 - e^{i2\pi(k-\ell)}}{1 - e^{i\frac{2\pi}{N}(k-\ell)}} = 0$$

which proves that this set of exponentials is an orthonormal set thus a linearly independent set of $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$. This set contains exactly N elements and vector space $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ is of dimension N thus this set of exponentials is an orthonormal basis of $\mathcal{F}_N(\mathbb{Z}, \mathbb{C})$. Therefore, any digital periodic signal x with period N can be written:

$$\forall n \in \mathbb{Z} \qquad x[n] = \sum_{k=0}^{N-1} \langle x, e^{i\frac{2\pi}{N}kn} \rangle_N e^{i\frac{2\pi}{N}kn}$$

with

$$\langle x, e^{i\frac{2\pi}{N}kn} \rangle_N = \frac{1}{N} \sum_{n=0}^{N-1} x[n] \overline{e^{i\frac{2\pi}{N}kn}} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi}{N}kn} = \frac{1}{N} X[k]$$ ∎

Proposition 2.4 (Plancherel's identity)

For any signal $x \in \mathcal{F}_N(\mathbb{Z}, \mathbb{C})$ with DFT X,

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

PROOF : As for Parseval's identity for Fourier series, this identity is an adaptation of Pythagorean theorem. Using the orthonormal basis of complex exponentials from the previous preposition, we can write:

$$\|x\|_N^2 = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 = \langle x, x\rangle_N = \left\langle \sum_{k=0}^{N-1}\frac{1}{N}X[k]e^{i\frac{2\pi}{N}kn}, \sum_{\ell=0}^{N-1}\frac{1}{N}X[\ell]e^{i\frac{2\pi}{N}\ell n}\right\rangle = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2$$

yielding the result. ■

Since x and X are both periodic with period N we can restrict both signals to samples $x[n]$ for $n \in [\![0, N-1]\!]$ and $X[k]$ for $k \in [\![0, N-1]\!]$. Thus we can consider signals x and X as vectors of $\mathbb{C}^N$ and the DFT as an mapping from $\mathcal{F}(\mathbb{Z}/N\mathbb{Z}, \mathbb{C})$ to $\mathcal{F}(\mathbb{Z}/N\mathbb{Z}, \mathbb{C})$, This is how signal processing softwares deal with signals. Setting $\omega = e^{-i\frac{2\pi}{N}}$, we can rewrite the definition of DFT matricially:

$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix} = A(\omega) \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix}$$

We can show by a computation that:

$$A(\omega)A(\omega^{-1}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \dots & \omega^{(N-1)^2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \dots & \omega^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)} & \omega^{-2(N-1)} & \dots & \omega^{-(N-1)^2} \end{pmatrix} = NI_N$$

which implies that the inverse of matrix $A(\omega)$ is $A(\omega)^{-1} = \frac{1}{N}A(\omega^{-1})$. Thus

$$\begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix} = \frac{1}{N}A(\omega^{-1}) \begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix}$$

which allows us to retrieve the inverse DFT.

2.3 Properties and example

Proposition 2.5

We consider digital signals of finite length N. The discrete Fourier transform satisfies the following properties:

(i) linearity: for any two signals x and y and two scalars α and β, $\mathcal{F}(\alpha x + \beta y) = \alpha\mathcal{F}(x) + \beta\mathcal{F}(y)$;

(ii) symmetry: for any signal x, if we denote $\tilde{x} : n \mapsto x[N-n]$, then $\mathcal{F}(\tilde{x}) = \widetilde{\mathcal{F}(x)} = \mathcal{F}(x^*)^*$;

(iii) circular convolution: for any two signals x and y, $\mathcal{F}(x \otimes y) = \mathcal{F}(x)\mathcal{F}(y)$;

(iv) time shift: for any $a \in [\![0, N-1]\!]$ and any signal x, $\mathcal{F}(\tau_a(x))[k] = e^{-i\frac{2\pi}{N}ka}\mathcal{F}(x)[k]$;

(v) multiplication: for any two signals x and y, $\mathcal{F}(xy) = \mathcal{F}(x) \otimes \mathcal{F}(y)$;

(vi) frequency shift for any $a \in [\![0, N-1]\!]$, $\mathcal{F}\left(xe^{i\frac{2\pi}{N}an}x\right) = \tau_a(\mathcal{F}(x))$

Proof : (i) We prove the linearity as for the other Fourier transforms.
(ii) We prove this property as for the DTFT using the change of variable $n \mapsto N-n$ instead of $n \mapsto -n$.
(iii) For any $k \in [\![0, N-1]\!]$,

$$\begin{aligned}\mathcal{F}(x \otimes y)[k] &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} x[m]e^{-i\frac{2\pi}{N}km}y[n-m]e^{-i\frac{2\pi}{N}k(n-m)} = \left(\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}\right)\left(\sum_{m=0}^{N-1} y[m]e^{-i\frac{2\pi}{N}km}\right)\\ &= \mathcal{F}(x)[k]\mathcal{F}(y)[k]\end{aligned}$$

(iv) For any $a \in [\![0, N-1]\!]$ and for any $k \in [\![0, N-1]\!]$,

$$\mathcal{F}(\tau_a(x))[k] = \sum_{n=0}^{N-1} x[n-a]e^{-i\frac{2\pi}{N}kn} = e^{-i\frac{2\pi}{N}ka}\sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn} = e^{-i\frac{2\pi}{N}ka}\mathcal{F}(x)[k]$$

(v) Let two signals x and y with respective DFTs X and Y. Let $Z = X \otimes Y$ and z the corresponding signal. Then for any $n \in [\![0, N-1]\!]$,

$$\begin{aligned}z[n] &= \frac{1}{N}\sum_{k=0}^{N-1}(X \otimes Y)[k]e^{i\frac{2\pi}{N}kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{\ell=0}^{N-1} X[\ell]Y[k-\ell]\right)e^{i\frac{2\pi}{N}kn}\\ &= \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{\ell=0}^{N-1} X[\ell]e^{i\frac{2\pi}{N}\ell n}Y[k-\ell]e^{i\frac{2\pi}{N}(k-\ell)n} = \frac{1}{N^2}\left(\sum_{k=0}^{N-1} X[k]e^{i\frac{2\pi}{N}kn}\right)\left(\sum_{\ell=0}^{N-1} Y[\ell]e^{i\frac{2\pi}{N}\ell n}\right)\\ &= x[n]y[n]\end{aligned}$$

Finally, $\mathcal{F}(xy) = \mathcal{F}(z) = \mathcal{F}(x) \otimes \mathcal{F}(y)$.
(vi) For any $k \in [\![0, N-1]\!]$,

$$\tau_a(\mathcal{F}(x))[k] = \mathcal{F}(x)[k-a] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}(k-a)n} = \sum_{n=0}^{N-1} x[n]e^{i\frac{2\pi}{N}an}e^{-i\frac{2\pi}{N}kn} = \mathcal{F}(xe^{i\frac{2\pi}{N}an})[k] \qquad \blacksquare$$

2.4 Fast Fourier transform

Consider a time-limited signal $x = (x[0], x[1], \dots, x[N-1])$ of length N. We recall that the DFT X of x is also of length N and is defined by:

$$\forall k \in [\![0, N-1]\!] \qquad X[k] = \sum_{n=0}^{N-1} x[n]e^{-i\frac{2\pi}{N}kn}$$

We express the complexity of computing this DFT in terms of the number of required multiplications of complex numbers. We assume that we have already access to the complex exponentials $e^{-i\frac{2\pi}{N}kn}$. In this case, each term of the sum requires one multiplication, thus the computation of one sample $X[k]$ requires N multiplications. Since there are N samples $X[k]$ to evaluate, the evaluation of the DFT X of length N requires N^2 multiplications.
A **Fast Fourier Transform** (FFT) algorithm uses recurrences in the definition of X to decrease the complexity of its evaluation. In this lecture, we present the **Cooley-Tukey algorithm** which expresses the DFT of a signal of length $2N$ as a function of the DFTs of two signals of length N. Thereby, consider a digital time-limited signal $x = (x[0], x[1], \dots, x[2N-1])$ of length $2N$,

and whose DFT X is also of length $2N$. We define the two following signals of length N: signal $\hat{x} = (x[0], x[2], \dots, x[2N-2])$ of even-index samples of x and signal $\tilde{x} = (x[1], x[3], \dots, x[2N-1])$ of odd-index samples of x. We denote $\widehat{X}$ et $\widetilde{X}$ their respective DFTs. For any $k \in [\![0, N-1]\!]$,

$$X[k] = \sum_{n=0}^{2N-1} x[n] e^{-i\frac{2\pi}{2N}kn} = \sum_{n=0}^{N-1} x[2n] e^{-i\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x[2n+1] e^{-i\frac{2\pi}{N}kn} e^{-i\frac{2\pi}{2N}k} = \sum_{n=0}^{N-1} \hat{x}[n] e^{-i\frac{2\pi}{N}kn} + e^{-i\frac{2\pi}{2N}k} \sum_{n=0}^{N-1} \tilde{x}[n] e^{-i\frac{2\pi}{N}kn}$$
$$= \widehat{X}[k] + e^{-i\frac{2\pi}{2N}k} \widetilde{X}[k]$$

This formula provides the N first values of the DFT X which is of lengh $2N$. Since DFTs $\widehat{X}[k]$ and $\widetilde{X}[k]$ are periodic with period N, we can find the remaining values of X by writing, for any $k \in [\![0, N-1]\!]$,

$$X[k+N] = \widehat{X}[k+N] + e^{-i\frac{2\pi}{2N}(k+N)} \widetilde{X}[k+N] = \widehat{X}[k] + e^{-i\frac{2\pi}{2N}k} e^{-i\pi} \widetilde{X}[k] = \widehat{X}[k] - e^{-i\frac{2\pi}{2N}k} \widetilde{X}[k]$$

Therefore, we have, for any $k \in [\![0, N-1]\!]$,

$$X[k] = \widehat{X}[k] + e^{-i\frac{2\pi}{2N}k} \widetilde{X}[k] \qquad \text{and} \qquad X[k+N] = \widehat{X}[k] - e^{-i\frac{2\pi}{2N}k} \widetilde{X}[k]$$

If $x = (x[0])$ is of length 1, then $X = (X[0])$ is also of length 1 and by the definition of the DFT, $X[0] = x[0]$. Therefore, we can write the following recursive algorithm to compute the DFT of a signal x whose length $N = 2^n$ is a power of 2.

Algorithm 1 Cooley-Tukey algorithm

procedure FFT(x)
    Input digital time-limited signal $x = (x[0], x[1], \dots, x[N-1])$ of length N ▷ Assume that $N = 2^n$ is a power of 2
    if $N = 1$ **then**
        $X = (X[0])$ is of length 1 and $X[0] \leftarrow x[0]$
    else
        Set $\hat{x} \leftarrow (x[0], x[2], \dots, x[N-2])$
        Set $\tilde{x} \leftarrow (x[1], x[3], \dots, x[N-1])$
        Compute $\widehat{X} \leftarrow \mathrm{FFT}(\hat{x})$
        Compute $\widetilde{X} \leftarrow \mathrm{FFT}(\tilde{x})$
        for $k \in [\![0, \frac{N}{2} - 1]\!]$ **do**
            $X[k] \leftarrow \widehat{X}[k] + e^{-i\frac{2\pi}{N}k} \widetilde{X}[k]$
            $X[k + N/2] = \widehat{X}[k] - e^{-i\frac{2\pi}{N}k} \widetilde{X}[k]$
        end for
    end if
    Return DFT $X = (X[0], X[1], \dots, X[N-1])$
end procedure

We can show that this algorithm requires the computation of $\dfrac{N}{2}\log_2(N)$ multiplications instead of N^2 for the direct evaluation.
Remark: This algorithm can be generalized to a signal of composite length $N = N_1 N_2$. In this case, the algorithm computes N_1 DFTs of sub-signals of lengths N_2.